Generalization in the Learning of Mathematics

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Abstract

Generalization is one of the fundamental activities in the learning of mathematics. The growing and improving mathematics is depended to applications of generalization from beginning until now. It seems that generalization needs to be introduced more among people who deal with mathematics. The main aim of this conceptual study is to explore the meanings of generalization from psychological and mathematical perspectives. In addition, the importance of generalization to support students in the learning of mathematical concepts is put forward.

Keywords: Generalization; mathematics, mathematical concepts, students' learning

INTRODUCTION

Generalization can be defined from a mathematical point of view which its means "looking for a bigger picture (Tall, 2011), consideration limited groups for consideration bigger groups or extending concepts to a bigger area to investigate about it" (Mason, Burton and Stacey, 2010). From a psychological viewpoint, generalization means "giving the same answers to similar stimulations" (Olson and Hergenhan, 2010). In other hands, generalization can be a general imagination of a topic that this imagination needs completely power of thinking (Piaget, 1980). Sriraman defines the generalization in the education of mathematics literature as the processes by which one drives or induces from particular cases (Yilmaz, Argun and Keskin, 2009). In this study generalization means looking for broader patterns and relationship and making connection in different levels of mathematical thinking (Stacy, 2006; Mason, Burton and Stacey, 2010).

According to Mason, Burton and Stacey (2010), using concepts in broader that can be useful for students to overcome difficulties is a kind of generalization. Using a generalization is necessary to have effective teaching and learning at all levels of mathematical learning (Mason et al., 2000). Mason and his colleagues (2000) have believed that main concepts of calculus are derivations and integrals; they also have shown the effectiveness of using the generalization to teach calculus among Malaysian students at UTM. Although using the generalization is useful in teaching calculus and its concepts such as derivations and integrals, but most of the teachers and students are forgetful to use it. In addition; using the generalization is useful because it is difficult for students to remember and use various methods in derivations and integrals, but the generalization covers this difficulty (Mason et al., 2000). Tall (2002a, 2008) and Mason and his colleagues (2010) believed that generalization is an important step in the mathematical thinking. In this conceptual paper, the generalization has been defined from different perspectives and its role and importance have been explained based on many researchers' works. It is necessary to explain the role of generalization in mathematical thinking form viewpoints of Tall and Mason, because generalization is a main step in mathematical thinking process.

GENERALIZATION FROM TALL

Tall (2002a, 2004b, 2012) asserted that mathematical thinking can support generalization in mathematic. Generalization's strategies are used in mathematic to show processes, which there are broader contexts in them and to help problem solver to know about those processes' products (Tall, 2002a). From Tall's point of view, the components of generalization can appear in three types; expansive, reconstruction and disjunctive.

First of all, the expansive as a generalization involves extending the existed information of learners without changing in their previous ideas. On the other hand, in expansive generalization, the new information should be close to the current information and should be in the same area. For example, in the vector space, at first student are taught on R and R^2 then educators add another component to plane (R^2) and introduce apace (R^3). Although the introducing of space is new for learners, there is no need to change their current ideas.

Furthermore, when a person extends a concept with a little change in his or her previous ideas or knowledge and tries to handle new one and currents, (s)he reconstructs his or her cognitive area of knowledge. This kind of change is named reconstruction generalization. To illustrate, in the example of vector space; when educators inset

R^4 which has contained width, Length, height and time, the component of time can be strange for students because they can see in daily life one- dimensional, two- dimensional and three- dimensional objects. The new dimension is unknown for students, thus to overcome this confusion; they should reconstruct their previous knowledge in vector space to convince accepting time as a fourth dimension in vector space. Bates (2006) also asserted that using mathematic in technology is kind of reconstructive generalization.

Next, disjunctive generating problems can be solved and analyzed in higher class of knowledge. To use of this kind of generating problem, students can solve new problems by adding numbers of disconnected pieces of information. Mason and his colleagues (2010) have covered this generating problem in the third stage (review) of his framework.

To sum up, the generalization from Tall's viewpoint has three components such as; expansive, reconstruction and disjunctive. In this study the three strategies of generalization that introduced by Tall (2002a), has been used among Iranian undergraduate students in derivations and integrals.

GENERALIZATION FROM MASON

It can be told that mathematics teaching is not anything unless problem solving (Polya, 1988). Thus, if a method or framework is belonged to problem solving, it can be belonged to mathematics teaching process and conversely. Mason (2012) has introduced two processes in his framework such as; specialization and generalization to teach concepts and to solve problems.

First of all, specialization which is firs operational process in teaching mathematics and using specialization systematically is important. Specialization in teaching of mathematic is using specific cases or examples (Mason, Burton, and Stacey (2010). In other word, specialization means turning to some examples to learn more about the main goal of concept or problem and these examples are specific and particular instances of more general situation in a concept (Mason, Burton, Stacey, 2010). Specializing determines teacher and student and a person who involved in teaching and learning process; what he/she knows about concept that past experiences can be used to help in this case, what he/she wants to do and what he/she can introduce. It should be considered that the specialization must be done systematically (Mason, Burton, and Stacey, 2010).

The reason that it should be used systematically is; trying to make a conjecture to reach to the main aims of concepts that is taught, for instance, by using systematically special examples of derivation in polynomials from linear to more degrees such as; 2th, 3th and 4th, the conjecture is made that refers to form the formula of derivation for polynomials of degree n, forming formula means trying to use symbols to enter to the next stage which named generalization Specialization can be appeared by drawing diagrams, constructing tables, making symbols and organization. In addition, Mason, Burton, and Stacey (2010) have mentioned that in problem solving specializing contains two phases; entry and attack, and generalization is contained attack and review.

Furthermore, generalization is the main next stage after specialization. When a person is faced new concepts or new problems after specializing and conjecturing, formulation is formed in his/her mind that is the beginning point to use generalization (Mason, Burton, and Stacey, 2010). Generalization activity can be extended from previous knowledge to new related concepts. To illustrate, in teaching of integral when students know that integral is the opposite of differentiation they can use their previous information in derivation that they have learned before and find answers of integral of other functions. Beside, students could use integral formulation and its rules to learn double integral by generalization. In generating three questions should be answered; what as conjecture, why as justification and where for more generating.

Watson and Mason (2006) have introduce many examples that were solved by using generalization and they have explained that why they choice those examples and what kind of properties were appeared in them. They have claimed that using generated examples by students can provide pedagogical tools to learn variety levels of mathematics (Zazkis and Leikin, 2007). Zazkis and Leikin (2007) have emphasizes that: "Watson and Mason focus on learner generated examples, a teaching strategy of asking learners to construct their own examples of mathematical objects under given constraints".

In conclusion, the perspective of Mason, Burton and Stacey (2010) is shown a regular flow of specialization, conjecturing, symbolization and generalization, of course he has emphasized to consider being systematic in specialization. Figure 1 can be useful for more information. It shows that between two main processes specialization and generalization, there are two adjuncts processes, but those adjuncts are important; conjecturing and symbolizing.
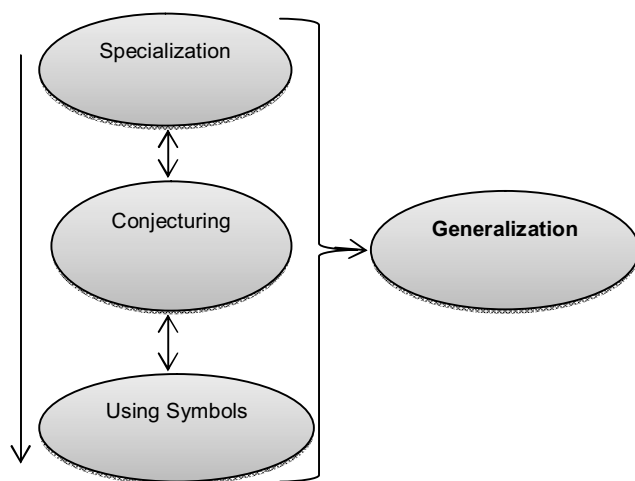
Figure 1: Relationship between generalization and specialization (Mason, Burton, and Stacey, 2010)

Mathematical thinking process which involves specialization, conjecturing and generalization has been applied in supporting difficulties in calculus and its concepts such as derivations and integrals. Yudariah (1997) has been used mathematical thinking process to change students' attitude toward calculus among Malaysian undergraduate students. Moreover, Roselainy (2008) has tried to use mathematical thinking to overcome students' difficulties in multivariable derivation and multiple integral. Both Yudariah and Roselainy have not used three worlds of mathematical thinking and they just used symbolic approach of presentation of derivations and integrals. However, Kashefi (2012) has applied computer to support undergraduate students to overcome their difficulties in the learning of partial derivations and multiple integrals by using mathematical worlds and mathematical thinking process.

Mathematical thinking is related to improving generalization skills for learning mathematics (Watson and Mason, 2006; Mason, Burton, and Stacey, 2010; Tall, 2012). Generalization as an important element of mathematical thinking and problem solving has potential to support students' learning in calculus (Tall, 2002a; 2012; Mason at el., 2010). However, this important element which can help students to overcome difficulties is feasible.

USING GENERALIZATION IN THE LEARNING OF MATHEMATIC
Many researchers have tried to find out the role of generalization in mathematics. Some people wanted to see how learners effort to generalize problems and theories in mathematics.

Tall (1993, 2010b, 2012), Limon (2001), Metaxas (2007) and Aghaee (2007) asserted that some difficulties to solve problems in derivations and integrals come back to inability of recalling prior knowledge. Generalization is a method to help covering distance between previous knowledge and new notions, because generalization can make link from previous knowledge to reach new related concepts (Stacey, 2006). For instance, if students refine their information and knowledge of function and limit to learn derivations and integrals, it will help them to learn these concepts better. It seems that there is a lack of connection and relationship between the pervious knowledge of calculus and news among students in learning process of derivations and integrals. To teach conceptually difficult notions of calculus such as derivations and integrals a cognitive distance between student's previous knowledge and new notions should be reduced with generating latters to news (Limon, 2001; Metaxas, 2007).

Tall (1993) stated that there are two impossible methods which students can use them to cope their difficulties in calculus; adapting the previous and new contrast knowledge structure by re-construction, and keeping conflicting new and old knowledge in two separate compartments. Nowadays, there are those difficulties yet because majority of students and even teachers use the second method which has not enough effects on students' learning of derivations and integrals (Tall, 1993; Tall, 2002b).

Most of teachers and students use generalization to learn derivations and integrals in symbolic approach rather than graphical (Tall, 2012). Some researchers (Roselainy, 2008; Kashefi, 2012) have tried to use generalization based on mathematical thinking, but they have used it in symbolic world. In addition, their application of

generalization was too superficial. However, some researchers (Cruz and Martinon1998; Becker and Rivera, 2005; Yesider and Akkoc, 2010) have used generalization deeply in algebraic and symbolic approach.

Becker and Rivera (2005) have tried to know which strategies of generalization are preferred to use in algebraic approach among students. According to Becker and Rivera (2005), in their prior research about 70% of participants were disabling to do generalization. In their main study about using generalization strategies in algebraic approach, Becker and Rivera (2005) selected 22 students, 11 male and 11 female. The students had to answer some questions about 20 or 30 minutes and to think with loud voice, it means they had to tell their thought loudly. Becker and Rivera tried to find that if they were using generalization in their solutions or no. the researchers find 23 strategies in students' solutions and their generalization strategies were as Table 1.

Table 1 The rate of using generalization in the study of Becker and Rivera (2005)

Category	Number
Able to generalize all parts	5
Able to generalize partially	4
Unable to generalize	13

Becker and Rivera (2005) have asserted that using generalization is the most important problem which should be introduced for students and they should be encouraged to use it. On the other hand, they could not use generalization in their algebraic questions. Also the researchers told that the generalization's strategies were just three in 23 such as; numerical, symbolical and graphical. The rate of using graphical generalization was less than others and students should be encouraged and educated to use this kind of generalization (Becker and Rivera, 2005). Later Yesider and Akkoc (2010) have shown that using algebraic generalization in more common among students.

Yesider and Akkoc (2010) have found that most of students in undergraduate level use generalization strategies in number patterns. Explicit, whole- object, chunking and recursive were four strategies that students prefer to use and all of these strategies are belong to number patterns. Yesider and Akkoc (2010) have defined the strategies as; (1) Explicit: A rule is constructed that allows for immediate calculation of any output value given a particular input value. (2) Whole-Object: Portion is used as a unit to construct a larger unit using multiples of the unit. (3) Chunking: Recursive pattern is built on by building a unit onto known values of the desired attribute. (4) Recursive: Relationship is described that occurs in the situation between consecutive values of the independent variable. However, they tested the using of these strategies in undergraduate students who use explicit more than others and the Recursive was used less than others. Yesider and Akkoc (2010) have suggested that student should learn to generate concepts graphically rather than symbolically or numerically. Although, Tall (2004b; 2008) had tried to use this kind of generalization in embodied world of mathematical thinking, but this utilization is for geometric concepts.

According to Tall (2004b), generalization in embodied world is mostly about geometric objects and in symbolic world is related to algebraic objects. In symbolic world generalization happens when the transition from operational procept to potential operation is occurred and this kind of generalization is reconstructive generalization (Tall, 2004b). To illustrate, when by operating on 2+3, 3+3, 4+3, and so on, in symbolic world the x+3 as potential operation reached with conjecturing, the reconstructive generalization has occurred (Tall, 2002a; Tall, 2004a). Also, Tall (1992, 1997, 2004b) emphasizes and credits to three kind of specialization (randomly, systematic and artfully) that happen before generalization in framework of Mason, Burton and Stacey (2010). Moreover, Tall beliefs that the systematic specialization which is happen in derivations and integrals can be more helpful to generate concepts.

Most of difficulties in embodied world are happen because figures and graphs which are used to show geometrical interpretation in derivations and integrals are largely separate from students' conceptual knowledge (Tall, 1993, 1997). Therefore, there is a logical reason to find a way that how this difficulty can be coped, and it seems using generalization in embodied world independently and transferring concepts from this world to symbolic world by generalization can help students to overcome their difficulties in university. Generalization can be imagined as a possible procedure to help students for using their information of high school calculus course in college calculus course. For doing better in this case of generalization it seems that re-construction generalization is

better and more useful (Tall, 1993, 2002a). Tall (2012) has asserted that experiences have effective role to do useful generalization, but he did not mention how can be this effectiveness.

The experiences could be supportive or problematic and impede, supportive to encourage students to generate and use previous learned concepts in new situations. When previous concept does as obstacle to learn new one, it can be named problematic and impede concept. Tall (2012) currently beliefs that there is a big gap between using mathematical notions in real world and study them in embodiment and symbolism. Most of mathematical rules that are known by symbols are not true in real world, but we have to use their symbolic aspect to find the exact solution of related problems. For instance, there is a big difference between the exact solution of distances relationship u and v object and image from a lens of focal length in mirror in formula such as 1/u+1/v=1/f with the answer that happen in real world by application of mirror. In other words, the relationships between u, v and f in real world are more than their relations which are calculated by functional and symbolic aspect and knowing this note can be help students to solve problems.

Generalization helps people to blend their experiences with useful knowledge to solve their problems in new conditions and lets them to be more creative. Tall (2012) tries to encourage students who are learning mathematics to generate mathematics by its application in daily life and practical activities: "mathematics must operate in a wide range of situations where particular embodiments give meanings that may not be applicable in other contexts". Mathematics should provide active and visual representation in embodiment world and it must be improve flexibility by using symbols to reach the last world of mathematic, formalism and proof. Tall (2012) added that generalization should be employed to solve problems in undergraduate level. The interesting thing is that by informing about the role of generalization in learning of calculus, Tall did not show how this generalization can be. Only Villers and Garner (2008) have demonstrated problem solving and proving via generalization; in that case they have shown just some examples that solved by using generalization methods same as Polya (1980, 1988) in his problem solving framework.

Villers and Garner (2008) have used strategies of generalization which were important for them to see their effects in the examples among students. First a few simple calculation examples were given and followed with some algebraic proof, there were two geometric examples that provided and finally two examples of complex numbers. To illustrate in following, a few of those example will be pointed to show how generating progress can be done. The models of algebraic and geometric examples were as below.

The following examples are algebraic proof that had been given in Villers and Garner's issue: (1) Prove that square of any odd number leaves reminder of 1 when divided 4. The solution is; $[(2n-1)]^2 = 4n^2 - 4n + 1 = 4(n^2 - n)$ +clearly if $4(n^2 - n)$ +divided to 4 the remainder will be 1.

One of geometrical examples was "two triangles are point perspective, if and if, they are also line perspective". Finally one of complex numbers examples had presented as below: show that $[r((\cos\theta) + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta)$. It is considerable that the geometrical generalization in this example has been done like algebraically. However, there is a big difference between algebraic and geometrical generalization. It can be seen clearly that they have used generalization in algebraic or symbolic mood of mathematics.

Algebraic generalization is coming back to generate in the degrees of polynomials (Becker and Rivera, 2005) or generating in symbolic and formal worlds (Tall, 2002a). Geometrical generalization is; extending properties or domination of shapes, graphs and vector space. For instance, generating and using the properties of equilateral triangle to prove some properties in square (Zaslavsky, 2007).

Zaslavsky and Zodik (2007) climbed that the knowledge for generating divided to mental and operational and students' background and experiences are important for them to use generalization. However, mental knowledge in their idea was mental imaginations, they found the results by analyzed of students answers. The questions in mental type were about students' mental abilities. In operational type student had been allowed to use figures and curves. Beside, mental knowledge is used to solve algebraic problems more often, but operational knowledge is used to solve problems that involve graph and curve. However, Honson (2004) has used operational generalization to prove some properties of hexagon by generation properties of triangle and called it geometrical generalization. Therefore, according to Zaslavsky (2007) and Zaslavsky and Zodik (2007), can be told that to learn concepts and solve problems in derivations and integrals using operational generalization is beneficial for graphs and curves. It should be noted that the example which used to learn and solve problems by generalization should be chosen logically and carefully (Watson and Mason, 2005).

Watson and Mason (2005) have given exercises to students to solve which were full of generalization. They have concluded there were too differences between common problems and the problems that had been generated, because students must generated the answer of the problems before. This kind of solution helped students to improve their abilities to solve more difficult questions by considering the generalization level by level; it means new

generating should be done based on preciouses by students (Watson and Mason, 2005). Also, in their study the quality of improving generalization power is not mentioned.

There are three levels of generalization in linear pattern in calculus (Cruz and Martinon, 1998). They have done empirical study on students to know what kind of generalization used. At the first video-recorded interviews were given to eleven students, then the students should look and pay to video carefully and answer to problems which were provided to them. Another group with 18 students from same age and same phase of education should judge the answers of first group. The judgment could not hint to first group about their solutions correct or incorrect, agree or disagree. After solution the responders should justify their solutions, in following their strategies about solution had been encouraged. Results have been analyzed and the researcher announced; there were three levels of generalization in those solutions, such as; (1) procedural activity: the recursive and iterative factors of linear pattern have been recognized by students. (2) Procedural understanding (local generalization): in this level the students should be able to establish local generalization, in other hand; she or he should be able to establish an invariant numerical sequence or picture by action performed in new problems, but the invariants in new problems could be different problem by problem. (3) Conceptual understanding (global generalization): at this level generalization has been used as strategy, in other story; the students could perform same solution and establish in new similar problems, using last solutions in new situations. The third kind of generalization that described by Cruz and Martinon (1998) can be seen in a work of Kabael (2011).

Kabael (2011) has studied about how can use generalization for generating single- variable to two- variable functions based on APOS (action, process, object, schema) theory. He has highlighted his belief in his study that teaching activities should support generalization of the functions and concepts which related to function such as limitation, derivations and integrals. To illustrate, he gave test in drawing graph regard to single- variable and two-variable to students and he chose six students to interview them about their solutions and drawing. Finally, he demonstrated that generalization could be happen in each step of APOS theory, however; he used the concepts of calculus in this study. In works of Cruz and Martinon (1998) and Kabael (2011) they only introduced the types of generalization in symbolic aspect which the main difficulties of students in focusing on this aspect.

In calculus most of activities are manipulation in symbolic world rather than problem- solving in this world of mathematical thinking (Tall, 1993; 1997). According to Tall (1993), successful students are able to be flexible in variety approach such as; symbolic, numeric and visual. But there is a disgusting of using visualization form of concepts and students prefer to work on numerical and symbolical approach regard to concepts in calculus and its concepts such as derivations and integrals. It means that there is a lack of manipulation in embodied world of mathematical thinking.

Yilmaz, Argun and keskin (2009) have studied about role of visualization in generalization processes, in their research the area was undergraduate students. They have focused on qualitative aspects of visualization when generalization was discovered by students. For analyzing and interpreting the finding of investigation, the researchers have used interviews and observations during writing activity; also they have tried to know about ability of students to formulae generalization using visualizations.

Yilmaz, Argun and keskin (2009) have concluded in their research, which was difficult for many students to see the relations between elements of problems and discovering generalization. Thus, the students rarely formulated generalization and the researchers have understood the students think their way through abstract situations by visualization. Beside, researchers have suggested that it would be better for mathematics teachers to use visualization in their teaching practices same as suggestion of Tall (2004a) and Mason and colleagues (2010) about using generalization in embodied world (Yilmaz, Argun and keskin, 2009).

Sriraman (2004) has described formulation of generalization in mathematics. He has examined whether it was possible for student to find and formulate generalization same as professional mathematicians. He has given four different problems to students and wanted to know about the experiences of students that they use for solving those problems, and then he has analyzed qualitatively and interpreted using imagination of uniframes based on Piaget theory about cognition and meta- cognition knowledge. At the result, Sriramam (2004) has written findings of his article which were firstly: problem selection is crucial if teacher wishes to invent problem- solving experiences that enable student to generalize. Secondly; generalizing behavior is dependent on being able to work over an extend time period on each problem. The implication was that solving and longitudinal study of structural similar problems are necessary to better understand higher level mathematical behaviors such as generalization. However, the ideas of Gardner (1997) have been near to Sriraman.

CONCLUSION

It seems that generalization is an important activity in the learning of mathematical concepts. Some students and teachers try to use it and some do not pay attention to it. According to many researchers which have been

explained before, generalization should be applied more in the learning of mathematics. In the previous sections, some advantage of using generalization has been presented.

Generally, the result shows that most of students and teachers in school and university do not pay attention to the role of generalization in their teaching process which can be observed in the lesson plan written by teachers themselves. Thus, there is a lack of generalization in the teaching process. Although generalization is the eleventh method for teaching mathematics, this method is often neglected in mathematics teaching and mathematics problem solving processes. It seems necessary to study in the future about the reasons of this negligence.

Based on previous studies, It seems axiomatic that if teaching skills of mathematics improved by using generalization, the quality and advantage of the learning product will be more useful. Thus in the curriculum of mathematics teachers' education, varieties of teaching methods should be considered. Generalization as an important property of mathematics should be used and taught frequently by mathematics teachers, because if teachers use this method in their teaching process, they will evoke learners to extend and transfer mental knowledge to practical knowledge such as; industry, health, socialism and so on.

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